

# SOJOURN TIMES AND THE FRAGILITY INDEX

TO APPEAR IN "STOCHASTIC PROCESSES AND THEIR APPLICATIONS"

MICHAEL FALK AND MARTIN HOFMANN

ABSTRACT. We investigate the sojourn time above a high threshold of a continuous stochastic process  $\mathbf{Y} = (Y_t)_{t \in [0,1]}$ . It turns out that the limit, as the threshold increases, of the expected sojourn time given that it is positive, exists if the copula process corresponding to  $\mathbf{Y}$  is in the functional domain of attraction of a max-stable process. This limit coincides with the limit of the fragility index corresponding to  $(Y_{i/n})_{1 \leq i \leq n}$  as  $n$  and the threshold increase.

If the process is in a certain neighborhood of a generalized Pareto process, then we can replace the constant threshold by a general threshold function and we can compute the asymptotic sojourn time distribution. A max-stable process is a prominent example. Given that there is an exceedance at  $t_0$  above the threshold, we can also compute the asymptotic distribution of the excursion time, which the process spends above the threshold function.

## 1. INTRODUCTION

Let  $\mathbf{Y} = (Y_t)_{t \in [0,1]}$  be a stochastic process with continuous sample paths, i.e.,  $\mathbf{Y}$  realizes in  $C[0,1]$ , and identical continuous marginal distribution functions (df)  $F$ , say. We investigate in this paper the sojourn time of  $\mathbf{Y}$  above a threshold  $s$

$$S(s) := \int_0^1 1(Y_t > s) dt,$$

under the condition that there is an exceedance, i.e.,  $S(s) > 0$ . Sojourn times of stochastic processes have been extensively studied in the literature, with emphasis on Gaussian processes and Markov random fields, we refer to Berman [3] and the literature given therein. A more general approach is the excursion random measure as investigated by Hsing and Leadbetter [11] for stationary processes. It is defined on sets  $E \subset \mathbb{R} \times (0, \infty)$  as the time which the process (suitably standardized) will spend in  $E$ . Different to that, we will investigate the sojourn time under the

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1991 *Mathematics Subject Classification.* Primary 60G70.

*Key words and phrases.* Sojourn time, fragility index, max-stable process, functional domain of attraction, copula process, generalized Pareto process, expected shortfall, sojourn time distribution, excursion time.

condition that the copula process  $\mathbf{C} := (F(Y_t))_{t \in [0,1]}$  corresponding to  $\mathbf{Y}$  is in the functional domain of attraction of a max-stable process  $\boldsymbol{\eta}$ , say.

Denote by  $N_s := \sum_{i=1}^n 1_{(s,\infty)}(Y_{i/n})$  the number of exceedances among  $(Y_{i/n})_{1 \leq i \leq n}$  above the threshold  $s$ . The *fragility index* (FI) corresponding to  $(Y_{i/n})_{1 \leq i \leq n}$  is defined as the asymptotic expectation of the number of exceedances given that there is at least one exceedance:

$$FI := \lim_{s \nearrow \omega(F)} E(N_s \mid N_s > 0),$$

where  $\omega(F) := \sup \{t \in \mathbb{R} : F(t) < 1\}$ . The FI was introduced in Geluk et al. [7] to measure the stability of a stochastic system. The system is called stable if  $FI = 1$ , otherwise it is called fragile. The collapse of a bank, symbolized by an exceedance, would be a typical example, illustrating the FI as a measure of joint stability among a portfolio of banks.

It turns out that the limit, as the threshold increases, of the expected sojourn time given that it is positive, exists if the copula process corresponding to  $\mathbf{Y}$  is in the functional domain of attraction of a max-stable process. This limit coincides with the limit of the FI corresponding to  $(Y_{i/n})_{1 \leq i \leq n}$  as  $n$  and the threshold increase.

For such processes, which are in a certain neighborhood of a generalized Pareto process (see Example 3.7), we can replace the constant threshold by a threshold function and we can compute the asymptotic sojourn time distribution above a high threshold function. A max-stable process is a prominent example. Given that there is an exceedance  $Y_{t_0} > s$  above the threshold  $s$  at  $t_0$ , we can also compute the asymptotic distribution of the remaining excursion time, that the process spends above the threshold function without cease.

This paper is organized as follows. In Section 2.1 we recall some mathematical framework from functional extreme value theory and provide basic definitions and tools. In particular we consider a functional domain of attraction approach for stochastic processes, which is more general than the usual one based on weak convergence. In Section 2.3 we apply the framework from Section 2.1 to copula processes and derive characterizations of the domain of attraction condition for copula processes. In Section 3 we use the results from Section 2.3 to compute the limit  $\lim_{s \nearrow \omega(F)} E(S(s) \mid S(s) > 0)$  as the threshold  $s$  increases of the mean sojourn time, conditional on the assumption that it is positive. We show that this limit coincides with the FI. Our tools enable also the computation of the expected shortfall. In Section 4 we replace the constant threshold by a threshold function and we compute the limit distribution of the sojourn time for those processes, which are in a certain neighborhood of a generalized Pareto process. Given that there is an exceedance at

$t_0$ , we compute in Section 5 the asymptotic distribution of the remaining excursion time that the process spends above a high threshold function.

To improve the readability of this paper we use bold face such as  $\boldsymbol{\xi}$ ,  $\boldsymbol{Y}$  for stochastic processes and default font  $f$ ,  $a_n$  etc. for non stochastic functions. Operations on functions such as  $\boldsymbol{\xi} < a$  or  $(\boldsymbol{\xi} - b_n)/a_n$  are meant componentwise. The usual abbreviations *df*, *fidis*, *iid* and *rv* for the terms *distribution function*, *finite dimensional distributions*, *independent and identically distributed* and *random variable*, respectively, are used.

## 2. DEFINITIONS AND PRELIMINARIES

**2.1. Max-stable Processes and the Functional D-Norm.** A *max-stable process* (MSP)  $\boldsymbol{\xi} = (\xi_t)_{t \in [0,1]}$  with realizations in  $C[0,1] := \{f : [0,1] \rightarrow \mathbb{R} : f \text{ continuous}\}$ , equipped with the sup-norm  $\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|$ , is a stochastic process with the characteristic property that its distribution is max-stable, i.e.,  $\boldsymbol{\xi}$  has the same distribution as  $\max_{1 \leq i \leq n} (\boldsymbol{\xi}_i - b_n)/a_n$  for independent copies  $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots$  of  $\boldsymbol{\xi}$  and some  $a_n, b_n \in C[0,1]$ ,  $a_n > 0$ ,  $n \in \mathbb{N}$  (cf. de Haan and Ferreira [9]).

We call a process  $\boldsymbol{\eta}$  with values in  $C^-[0,1] := \{f \in C[0,1] : f < 0\}$  a *standard MSP*, if it is a MSP with standard negative exponential (one-dimensional) margins,  $P(\eta_t \leq x) = \exp(x)$ ,  $x \leq 0$ ,  $t \in [0,1]$ .

In what follows  $\bar{C}^-[0,1] := \{f \in C[0,1] : f \leq 0\}$  denotes the set of all continuous function on  $[0,1]$  that do not attain positive values.

The following characterization is essentially due to Giné et al. [8]; we refer also to Aulbach et al. [2].

**Proposition 2.1.** *A process  $\boldsymbol{\eta}$  with realizations in  $C^-[0,1]$  is a standard MSP if, and only if there exists a number  $m \geq 1$  and a stochastic process  $\boldsymbol{Z}$  in  $\bar{C}^+[0,1] := \{f \in C[0,1] : f \geq 0\}$  with the properties*

$$(1) \quad \max_{t \in [0,1]} Z_t = m, \quad E(Z_t) = 1, \quad t \in [0,1],$$

*such that for compact subsets  $K_1, \dots, K_d$  of  $[0,1]$  and  $x_1, \dots, x_d \leq 0$ ,  $d \in \mathbb{N}$ ,*

$$(2) \quad P(\eta_t \leq x_j, t \in K_j, 1 \leq j \leq d) = \exp \left( -E \left( \max_{1 \leq j \leq d} |x_j| \max_{t \in K_j} Z_t \right) \right).$$

*Conversely, every stochastic process  $\boldsymbol{Z}$  with realizations in  $\bar{C}^+[0,1]$  satisfying (1) gives rise to a standard MSP. The connection is via (2). We call  $\boldsymbol{Z}$  generator of  $\boldsymbol{\eta}$ .*

According to de Haan and Ferreira [9, Corollary 9.4.5] the condition  $\max_{t \in [0,1]} Z_t = m$  in (1) can be replaced by the condition  $E(\max_{t \in [0,1]} Z_t) < \infty$ . The number

$m = E(\max_{t \in [0,1]} Z_t)$  is uniquely determined, see Remark 3.3. Therefore, we call  $m$  the *generator constant* of  $\boldsymbol{\eta}$ .

The preceding characterization implies in particular that the fidis of  $\boldsymbol{\eta}$  are multivariate negative EVD with standard negative exponential margins: We have for  $0 \leq t_1 < t_2 \cdots < t_d \leq 1$

$$(3) \quad -\log(G_{t_1, \dots, t_d})(\mathbf{x}) = E\left(\max_{1 \leq i \leq d} (|x_i| Z_{t_i})\right) =: \|\mathbf{x}\|_{D_{t_1, \dots, t_d}}, \quad \mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d,$$

where  $\|\cdot\|_{D_{t_1, \dots, t_d}}$  is a  $D$ -norm on  $\mathbb{R}^d$  (cf. Falk et al. [5]).

Let  $E[0, 1]$  be the set of all bounded real-valued functions on  $[0, 1]$  which are discontinuous at a finite set of points. Moreover, denote by  $\bar{E}^-[0, 1]$  the set of those functions in  $E[0, 1]$  which do not attain positive values.

For a generator process  $\mathbf{Z}$  in  $\bar{C}^+[0, 1]$  as in Proposition 2.1 and all  $f \in E[0, 1]$  set

$$\|f\|_D := E\left(\sup_{t \in [0, 1]} (|f(t)| Z_t)\right).$$

Obviously,  $\|\cdot\|_D$  defines a norm on  $E[0, 1]$ , called a  $D$ -norm with generator  $\mathbf{Z}$ ; see Aulbach et al. [2] for further details.

The following result is established in Aulbach et al. [2].

**Lemma 2.2.** *Let  $\boldsymbol{\eta}$  be a standard MSP with generator  $\mathbf{Z}$ . Then we have for each  $f \in \bar{E}^-[0, 1]$*

$$(4) \quad P(\boldsymbol{\eta} \leq f) = \exp(-\|f\|_D) = \exp\left(-E\left(\sup_{t \in [0, 1]} (|f(t)| Z_t)\right)\right).$$

*Conversely, if there is some  $\mathbf{Z}$  with properties (1) and some  $\boldsymbol{\eta} \in C^-[0, 1]$  which satisfies (4), then  $\boldsymbol{\eta}$  is standard max-stable with generator  $\mathbf{Z}$ .*

The representation  $P(\boldsymbol{\eta} \leq f) = \exp(-\|f\|_D)$ ,  $f \in \bar{C}^-[0, 1]$ , of a standard MSP is in complete accordance with the df of a multivariate EVD with standard negative exponential margins via a  $D$ -norm on  $\mathbb{R}^d$  as developed in Falk et al. [5, Section 4.4].

Note that for  $d \in \mathbb{N}$ ,  $0 \leq t_1 < \cdots < t_d \leq 1$  and  $\mathbf{x} = (x_1, \dots, x_d) \in (-\infty, 0]^d$ , the function

$$f(t) = \sum_{i=1}^d x_i 1_{\{t_i\}}(t), \quad t \in [0, 1],$$

is an element of  $\bar{E}^-[0, 1]$  with the property

$$P(\boldsymbol{\eta} \leq f) = \exp\left(-\|\mathbf{x}\|_{D_{t_1, \dots, t_d}}\right).$$

So representation (4) incorporates all fidis of  $\boldsymbol{\eta}$ . This is one of the reasons, why we favor a MSP  $\boldsymbol{\eta}$  with standard negative exponential margins, whereas de Haan and Ferreira [9], for instance, consider a continuous MSP  $\boldsymbol{\xi} = (\xi_t)_{t \in [0, 1]}$  with standard

Fréchet margins  $P(\xi_t \leq x) = \exp(-x^{-1})$ ,  $x > 0$ , called *simple* MSP. Actually, these are dual approaches, as we have

$$\xi = -\frac{1}{\eta} \text{ and } \eta = -\frac{1}{\xi},$$

taken pointwise (see Aulbach et al. [2]). A simple MSP satisfies for  $g : [0, 1] \rightarrow (0, \infty)$  with  $\tilde{f} := -1/g \in \bar{E}^-[0, 1]$ , consequently,

$$P(\xi \leq g) = P\left(\eta \leq -\frac{1}{g}\right) = \exp\left(-\left\|\frac{1}{g}\right\|_D\right),$$

but, different to  $\eta$  we do not obtain the fidis of  $\xi$  by a suitable choice of  $g$ .

Just like in the uni- or multivariate case, we might consider

$$H(f) := P(\mathbf{Y} \leq f), \quad f \in \bar{E}^-[0, 1],$$

as the df of a stochastic process  $\mathbf{Y}$  in  $\bar{C}^-[0, 1]$ .

**2.2. Functional Domain of Attraction.** According to Aulbach et al. [2] we say that a stochastic process  $\mathbf{Y}$  in  $C[0, 1]$  is *in the functional domain of attraction of a standard MSP  $\eta$* , denoted by  $\mathbf{Y} \in \mathcal{D}(\eta)$ , if there are functions  $a_n \in C^+[0, 1]$ ,  $b_n \in C[0, 1]$ ,  $n \in \mathbb{N}$ , such that

$$(5) \quad \lim_{n \rightarrow \infty} P\left(\frac{\mathbf{Y} - b_n}{a_n} \leq f\right)^n = P(\eta \leq f) = \exp(-\|f\|_D)$$

for any  $f \in \bar{E}^-[0, 1]$ . This is equivalent to

$$(6) \quad \lim_{n \rightarrow \infty} P\left(\max_{1 \leq i \leq n} \frac{\mathbf{Y}_i - b_n}{a_n} \leq f\right) = P(\eta \leq f)$$

for any  $f \in \bar{E}^-[0, 1]$ , where  $\mathbf{Y}_1, \mathbf{Y}_2, \dots$  are independent copies of  $\mathbf{Y}$ .

There should be no risk of confusion with the notation of domain of attraction in the sense of weak convergence of stochastic processes as investigated in de Haan and Lin [10]. But to distinguish between these two approaches we will consistently speak of *functional* domain of attraction in this paper, when the above definition is meant. Actually, this definition of domain of attraction is less restrictive as the next lemma shows; it is established in Aulbach et al. [2].

**Lemma 2.3.** *Suppose that  $\mathbf{Y}$  is a continuous process in  $\bar{C}^-[0, 1]$ . If the sequence of continuous processes  $\mathbf{X}_n := \max_{1 \leq i \leq n} ((\mathbf{Y}_i - b_n)/a_n)$  converges weakly in  $\bar{C}^-[0, 1]$ , equipped with the sup-norm  $\|\cdot\|_\infty$ , to the standard MSP  $\eta$ , then  $\mathbf{Y} \in \mathcal{D}(\eta)$  in the sense of condition (5).*

Note that the reverse implication in the preceding does not hold, i.e., convergence in the sense of condition (5) is strictly weaker than weak convergence in  $C[0, 1]$ . One can also show that (5) implies hypoconvergence of the normalized maximum process in the sense of Molchanov [12, Chapter 5, Section 3.1].

**2.3. Domain of Attraction for Copula Processes.** The sojourn time distribution of a stochastic process  $(Y_t)_{t \in [0,1]}$  with identical continuous univariate marginal df  $F$  does not depend on this marginal df but on the corresponding copula process. This is immediate from the equality  $\int_0^1 1(Y_t > s) dt = \int_0^1 1(U_t > F(s)) dt$ , where  $U_t := F(Y_t)$  is uniformly on  $(0, 1)$  distributed for each  $t \in [0, 1]$ . We, therefore, recall in this section results for copula processes established in Aulbach et al. [2].

Let  $\mathbf{Y} = (Y_t)_{t \in [0,1]}$  be a continuous stochastic process with identical continuous marginal df  $F$ . Set

$$\mathbf{U} = (U_t)_{t \in [0,1]} = (F(Y_t))_{t \in [0,1]},$$

which is the *copula process* corresponding to  $\mathbf{Y}$ .

We conclude from de Haan and Lin [10] that the process  $\mathbf{Y}$  is in the domain of attraction of a MSP if, and only if each  $Y_t$  is in the domain of attraction of a univariate extreme value distribution together with the condition that the copula process converges in distribution to a standard MSP  $\boldsymbol{\eta}$ , that is

$$\left( \max_{1 \leq i \leq n} n(U_t^{(i)} - 1) \right)_{t \in [0,1]} \rightarrow_D \boldsymbol{\eta}$$

in  $C[0, 1]$ , where  $\mathbf{U}^{(i)}$ ,  $i \in \mathbb{N}$ , are independent copies of  $\mathbf{U}$ . Note that the univariate margins determine the norming constants, so the norming functions can be chosen as the constant functions  $a_n = 1/n$ ,  $b_n = 1$ ,  $n \in \mathbb{N}$ . Lemma 2.3 implies that  $\mathbf{U}$  is in the functional domain of attraction of  $\boldsymbol{\eta}$ .

Suppose that the rv  $(Y_{i/d})_{i=1}^d$  is in the ordinary domain of attraction of a multivariate EVD (see, for instance, Falk et al. [5, Section 5.2]). Then we know from Aulbach et al. [1] that the copula  $C_d$  corresponding to the rv  $(Y_{i/d})_{i=1}^d$  satisfies the equation

$$(7) \quad C_d(\mathbf{y}) = 1 - \|\mathbf{1} - \mathbf{y}\|_{D_d} + o(\|\mathbf{1} - \mathbf{y}\|_\infty),$$

as  $\|\mathbf{1} - \mathbf{y}\|_\infty \rightarrow 0$ , uniformly in  $\mathbf{y} \in [0, 1]^d$ , where the  $D$ -norm is given by

$$\|\mathbf{x}\|_{D_d} = E \left( \max_{1 \leq i \leq d} (|x_i| Z_{i/d}) \right), \quad \mathbf{x} \in \mathbb{R}^d.$$

The following analogous result for the functional domain of attraction was established in Aulbach et al. [2].

**Proposition 2.4.** *Suppose that  $\mathbf{U}$  with realizations in  $\bar{C}^+[0, 1]$  is a copula process. The following equivalences hold:*

$\mathbf{U} \in \mathcal{D}(\boldsymbol{\eta})$  in the sense of condition (5)

$$\iff P \left( \mathbf{U} - \mathbf{1} \leq \frac{\mathbf{f}}{n} \right) = 1 - \left\| \frac{\mathbf{f}}{n} \right\|_D + o \left( \frac{1}{n} \right), \quad \mathbf{f} \in \bar{E}^-[0, 1], \text{ as } n \rightarrow \infty,$$

$$(8) \quad \Longleftrightarrow P(\mathbf{U} - 1 \leq |c| f) = 1 + c \|f\|_D + o(c), \quad f \in \bar{E}^- [0, 1], \text{ as } c \uparrow 0,$$

Note that condition (8) holds if

$$(8') \quad P(\mathbf{U} - 1 \leq g) = 1 - \|g\|_D + o(\|g\|_\infty)$$

as  $\|g\|_\infty \rightarrow 0$ , uniformly for all  $g \in \bar{E}^- [0, 1]$  with  $\|g\|_\infty \leq 1$ . It is an open problem whether (8') and (8) are, actually, equivalent conditions.

### 3. SOJOURN TIMES AND THE FRAGILITY INDEX

Let  $\mathbf{Y} = (Y_t)_{t \in [0,1]}$  be a continuous stochastic process with identical continuous marginal df  $F$ . We investigate in this section the mean of the sojourn time of  $\mathbf{Y}$  above a threshold  $s$

$$S(s) = \int_0^1 1(Y_t > s) dt,$$

under the condition that there is an exceedance, i.e.,  $S(s) > 0$ . In particular we establish its asymptotic equality with the limit of the FI corresponding to  $(Y_{i/n})_{1 \leq i \leq n}$ .

Before we present the main results of this section we need some auxiliary results. Put for  $n \in \mathbb{N}$

$$S_n(s) := \frac{1}{n} \sum_{i=1}^n 1(Y_{i/n} > s),$$

which is a Riemann sum of the integral  $S(s)$ . We have

$$S_n(s) \rightarrow_{n \rightarrow \infty} S(s)$$

and, thus,

$$P(S_n(s) \leq x) \rightarrow_{n \rightarrow \infty} P(S(s) \leq x)$$

for each  $x \geq 0$  such that  $P(S(s) = x) = 0$ . As a consequence we obtain

$$\begin{aligned} P(S_n(s) \leq x \mid S_n(s) > 0) &= \frac{P(0 < S_n(s) \leq x)}{P(S_n(s) > 0)} \\ &\rightarrow_{n \rightarrow \infty} \frac{P(0 < S(s) \leq x)}{P(S(s) > 0)} \\ &= P(S(s) \leq x \mid S(s) > 0) \end{aligned}$$

for each such  $x > 0$ . This conclusion requires the following argument.

**Lemma 3.1.** *We have*

$$P(S_n(s) = 0) \rightarrow_{n \rightarrow \infty} P(S(s) = 0),$$

*Proof.* We have

$$P(S_n(s) = 0) \leq P(S_n(s) \leq \varepsilon) \xrightarrow{n \rightarrow \infty} P(S(s) \leq \varepsilon) = P(S(s) = 0) + \delta,$$

where  $\varepsilon, \delta > 0$  can be made arbitrarily small. This implies  $\limsup_{n \rightarrow \infty} P(S_n(s) = 0) \leq P(S(s) = 0)$ . We have, on the other hand,

$$P(S(s) = 0) = P\left(\bigcap_{n \in \mathbb{N}} \{S_n(s) = 0\}\right) \leq \liminf_{n \rightarrow \infty} P(S_n(s) = 0),$$

which implies the assertion.  $\square$

We have

$$\begin{aligned} S_n(s) &= \frac{1}{n} \sum_{i=1}^n 1(F(Y_{i/n}) > F(s)) \\ &= \frac{1}{n} \sum_{i=1}^n 1(U_{i/n} > c) \end{aligned}$$

almost surely, where  $c := F(s)$ .

Note that

$$\begin{aligned} FI_n(s) &:= E(nS_n(s) \mid S_n(s) > 0) \\ &= E\left(\sum_{i=1}^n 1(U_{i/n} > c) \mid S_n(s) > 0\right) \\ &= \sum_{i=1}^n P(U_{i/n} > c \mid S_n(s) > 0) \\ &= \sum_{i=1}^n \frac{P(U_{i/n} > c)}{P(S_n(s) > 0)} \\ &= n \frac{1 - c}{1 - P(S_n(s) = 0)} \end{aligned}$$

is the FI of level  $s$  corresponding to  $Y_{i/n}$ ,  $1 \leq i \leq n$ . For an extensive investigation and extension of the FI we refer to Falk and Tichy [6]. The following theorem is the first main result of this section.

**Theorem 3.2.** *Let  $\mathbf{Y}$  be a stochastic process in  $C[0, 1]$  with identical continuous marginal df  $F$ . Suppose that the copula process  $\mathbf{U} = (F(Y_t))_{t \in [0, 1]}$  corresponding to  $\mathbf{Y}$  is in the functional domain of attraction of a MSP  $\boldsymbol{\eta}$  with generator constant  $m \geq 1$  as in Proposition 2.1. Then we have*

$$\lim_{n \rightarrow \infty} \lim_{s \nearrow \omega(F)} \frac{FI_n(s)}{n} = \lim_{s \nearrow \omega(F)} \lim_{n \rightarrow \infty} \frac{FI_n(s)}{n} = \lim_{s \nearrow \omega(F)} E(S(s) \mid S(s) > 0) = \frac{1}{m}.$$



*Proof.* Expansion (7) implies for  $n \in \mathbb{N}$

$$\begin{aligned}
& P(S_n(s) > 0) \\
&= 1 - P\left(\sum_{i=1}^n 1(U_{i/n} > c) = 0\right) \\
&= 1 - P(U_{i/n} \leq c, 1 \leq i \leq n) \\
&= 1 - C_n(c, \dots, c) \\
&= (1 - c) \|(1, \dots, 1)\|_{D_n} + o\left((1 - c) \|(1, \dots, 1)\|_{D_n}\right) \\
&= (1 - c)E\left(\max_{1 \leq i \leq n} Z_{i/n}\right) + o\left((1 - c)E\left(\max_{1 \leq i \leq n} Z_{i/n}\right)\right)
\end{aligned}$$

as  $c \uparrow 1$  and, thus,

$$\begin{aligned}
\frac{FI_n(s)}{n} &= \frac{1 - c}{P(S_n(s) > 0)} \\
&= \frac{1}{E\left(\max_{1 \leq i \leq n} Z_{i/n}\right) + o\left(E\left(\max_{1 \leq i \leq n} Z_{i/n}\right)\right)}
\end{aligned}$$

as  $c \uparrow 1$ . We, thus, obtain

$$\lim_{n \rightarrow \infty} \lim_{s \nearrow \omega(F)} \frac{FI_n(s)}{n} = \lim_{n \rightarrow \infty} \frac{1}{E\left(\max_{1 \leq i \leq n} Z_{i/n}\right)} = \frac{1}{E\left(\max_{0 \leq t \leq 1} Z_t\right)} = \frac{1}{m}.$$

We have, on the other hand,

$$\lim_{n \rightarrow \infty} \frac{FI_n(s)}{n} = \lim_{n \rightarrow \infty} \frac{1 - c}{1 - P(S_n(s) = 0)} = \frac{1 - c}{1 - P(S(s) = 0)}.$$

Since  $\mathbf{U} \in \mathcal{D}(\boldsymbol{\eta})$ , we obtain from the equivalent condition (8)

$$\begin{aligned}
\lim_{s \nearrow \omega(F)} \lim_{n \rightarrow \infty} \frac{FI_n(s)}{n} &= \lim_{s \nearrow \omega(F)} \frac{1 - c}{1 - P(S(s) = 0)} \\
&= \lim_{s \nearrow \omega(F)} \frac{1 - c}{1 - P(\mathbf{Y} \leq s)} \\
&= \lim_{s \nearrow \omega(F)} \frac{1 - c}{1 - P(\mathbf{U} \leq c)} \\
&= \lim_{s \nearrow \omega(F)} \frac{1 - c}{1 - (1 - (1 - c) \|\mathbf{1}\|_D + o(1 - c))} \\
&= \frac{1}{\|\mathbf{1}\|_D} \\
&= \frac{1}{E\left(\max_{0 \leq t \leq 1} Z_t\right)} \\
&= \frac{1}{m},
\end{aligned}$$

where 1 is the constant function on  $[0, 1]$ . Moreover, by the dominated convergence theorem

$$\begin{aligned} \frac{FI_n(s)}{n} &= E(S_n(s) \mid S_n(s) > 0) \\ &= \frac{E(S_n(s))}{P(S_n(s) > 0)} \\ &\xrightarrow{n \rightarrow \infty} \frac{E(S(s))}{P(S(s) > 0)} \\ &= E(S(s) \mid S(s) > 0). \end{aligned}$$

□

REMARK 3.3. While the generator  $\mathbf{Z}$  of a standard MSP  $\boldsymbol{\eta}$  is in general not uniquely determined, the generator constant  $m = E(\max_{t \in [0,1]} Z_t) = \|1\|_D$  is.

REMARK 3.4. Under the conditions of Theorem 3.2 we have

$$P(S(s) > 0) = (1 - c)m + o(1 - c) \quad \text{as } c \nearrow 1 \quad \text{and} \quad E(S(s)) = 1 - F(s).$$

To apply the preceding result to generalized Pareto processes defined below, we add an extension of Theorem 3.2. It is shown by repeating the preceding arguments.

We call a copula process  $\mathbf{U} = (U_t)_{t \in [0,1]}$  (upper) *tail continuous*, if the process  $\mathbf{U}_{c_0} := (\max(c_0, U_t))_{t \in [0,1]}$  is a.s. continuous for some  $c_0 < 1$ . Note that in this case  $\mathbf{U}_c$  is a.s. continuous for each  $c \geq c_0$ .

A stochastic process  $\mathbf{Y} = (Y_t)_{t \in [0,1]}$  is said to have ultimately identical and continuous marginal df  $F_t$ ,  $t \in [0, 1]$ , if  $F_t(x) = F_s(x)$ ,  $0 \leq s, t \leq 1$ ,  $x \geq x_0$  with  $F_1(x_0) < 1$ , and  $F_1(x)$  is continuous for  $x \geq x_0$ .

**Theorem 3.5.** *Let  $\mathbf{Y} = (Y_t)_{t \in [0,1]}$  be a stochastic process with ultimately identical and continuous marginal df. Suppose that the copula process pertaining to  $\mathbf{Y}$  is tail continuous and that it is in the functional domain of attraction of a MSP  $\boldsymbol{\eta}$ , whose finite dimensional marginal distributions are given by*

$$G_{t_1, \dots, t_d}(\mathbf{x}) = \exp \left( -E \left( \max_{1 \leq i \leq d} |x_i| Z_{t_i} \right) \right),$$

$0 \leq t_1 < \dots < t_d \leq 1$ ,  $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$ ,  $d \in \mathbb{N}$ . We require that the stochastic process  $\mathbf{Z} = (Z_t)_{t \in [0,1]}$  is a.s. continuous and that its components satisfy  $0 \leq Z_t \leq m$  a.s.,  $E(Z_t) = 1$ ,  $t \in [0, 1]$ , for some  $m \geq 1$ . Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \lim_{s \nearrow \omega(F)} \frac{FI_n(s)}{n} &= \lim_{s \nearrow \omega(F)} \lim_{n \rightarrow \infty} \frac{FI_n(s)}{n} \\ &= \lim_{s \nearrow \omega(F)} E(S(s) \mid S(s) > 0) \end{aligned}$$

$$= \frac{1}{E(\max_{0 \leq t \leq 1} Z_t)}.$$

EXAMPLE 3.6. Consider the  $d$ -dimensional EVD  $G(\mathbf{x}) = \exp(-\|\mathbf{x}\|_p)$ ,  $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$ ,  $d \geq 2$ , where the  $D$ -norm is the usual  $p$ -norm  $\|\mathbf{x}\|_D = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p} = \|\mathbf{x}\|_p$ ,  $\mathbf{x} \in \mathbb{R}^d$ , with  $1 \leq p \leq \infty$ . This is known as the Gumbel-Hougaard or logistic model. The case  $p = \infty$  yields the maximum-norm  $\|\mathbf{x}\|_\infty$ . Let the rv  $(Z_1, \dots, Z_d)$  be a generator of  $\|\cdot\|_p$ , i.e.,  $0 \leq Z_i \leq c$  a.s.,  $E(Z_i) = 1$ ,  $1 \leq i \leq d$  with some  $c \geq 1$ , and  $\|\mathbf{x}\|_p = E(\max_{1 \leq i \leq d}(|x_i| Z_i))$ ,  $\mathbf{x} \in \mathbb{R}^d$ . The rv  $(Z_1, \dots, Z_d)$  can be extended by linear interpolation to a generator  $\mathbf{Z} = (Z_t)_{t \in [0,1]}$  of a standard MSP  $\boldsymbol{\eta}$ : Put for  $i = 1, \dots, d-1$

$$Z_{(1-\vartheta)\frac{i-1}{d-1} + \vartheta\frac{i}{d-1}} := (1-\vartheta)Z_{i-1} + \vartheta Z_i, \quad 0 \leq \vartheta \leq 1,$$

which yields a continuous generator  $\mathbf{Z} = (Z_t)_{t \in [0,1]}$ . In this case we have

$$\frac{1}{E(\max_{0 \leq t \leq 1} Z_t)} = \frac{1}{E(\max_{1 \leq i \leq d} Z_i)} = \frac{1}{\|(1, \dots, 1)\|_p} = \frac{1}{d^{1/p}},$$

i.e., the generator constant is  $d^{1/p}$ .

Note that a standard MSP  $\boldsymbol{\eta}$ , whose finite dimensional marginal distributions  $G_{t_1, \dots, t_d}$  are for each set  $0 \leq t_1 < t_2 < \dots < t_d \leq 1$  and each  $d \geq 1$  given by  $G_{t_1, \dots, t_d}(\mathbf{x}) = \exp(-\|\mathbf{x}\|_p)$ ,  $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$ , does not exist for  $p \in [1, \infty)$ . This follows from the fact that in this case the generator constant would be infinite. In case  $p = \infty$ , which is the case of complete dependence, one can choose  $\boldsymbol{\eta} = (\eta_t)_{t \in [0,1]}$  with  $\eta_t := \eta$ ,  $t \in [0, 1]$ , where  $\eta$  is a rv with standard negative exponential distribution. As a generator one can choose the constant function  $Z_t = 1$ ,  $t \in [0, 1]$ .

EXAMPLE 3.7 (Generalized Pareto Process (GPP)). Let  $\mathbf{Z} = (Z_t)_{t \in [0,1]}$  in  $\bar{C}^+[0, 1]$  with  $0 \leq Z_t \leq m$  a.s.,  $E(Z_t) = 1$ ,  $t \in [0, 1]$ , for some  $m \geq 1$ , and let  $U$  be a rv that is uniformly on  $(0, 1)$  distributed and which is independent of  $\mathbf{Z}$ . Then the process

$$\mathbf{Y} := \frac{1}{U} \mathbf{Z}$$

with values in  $\bar{C}^+[0, 1]$  is an example of a *generalized Pareto process (GPP)* (cf. Buishand et al. [4]), as its univariate margins are (in its upper tails) standard Pareto distributions:

$$\begin{aligned} F_t(x) &= P(Z_t \leq xU) \\ &= \int_0^m P\left(\frac{z}{x} < U\right) (P * Z_t)(dz) \\ &= 1 - \frac{1}{x} E(Z_t) \\ &= 1 - \frac{1}{x}, \quad x \geq m, t \in [0, 1]. \end{aligned}$$

Here  $P * Z_t$  denotes the probability measure induced by  $Z_t$ , i.e.,  $(P * Z_t)(B) = P(Z_t \in B)$  for each  $B$  in the Borel- $\sigma$  field of  $\mathbb{R}$ .

We have, moreover, by Fubini's theorem for all  $f \in \bar{E}^-[0, 1]$  with  $\|f\|_\infty \leq 1/m$

$$P\left(-\frac{1}{\mathbf{Y}} \leq f\right) = 1 - \|f\|_D,$$

i.e., the GPP  $\mathbf{V} := (\max(-1/Y_t, M))_{0 \leq t \leq 1} = (\max(-U/Z_t, M))_{0 \leq t \leq 1}$ , with an arbitrary constant  $M < 0$ , has the property that its df is in its upper tail equal to

$$W(f) := P(\mathbf{V} \leq f) = 1 + \log(G(f)), \quad f \in \bar{E}^-[0, 1], \|f\|_\infty \leq 1/m,$$

where  $G(f) = P(\boldsymbol{\eta} \leq f)$  is the df of the MSP  $\boldsymbol{\eta}$  with  $D$ -norm  $\|\cdot\|_D$  and generator  $\mathbf{Z}$  (cf. Aulbach et al. [2, Section 4]).

The preceding representation of the upper tail of the df of a GPP  $\mathbf{V}$  in terms of  $1 + \log(G)$  is in complete accordance with the unit- and multivariate case (see, for example, Falk et al. [5, Chapter 5]).

We call in general a stochastic process  $\mathbf{V}$  in  $\bar{C}^-[0, 1]$  a *standard* GPP, if there is  $\varepsilon_0 > 0$ ,  $M < 0$  with  $P(\mathbf{V} \leq f) = P((\max(-U/Z_t, M))_{0 \leq t \leq 1} \leq f)$  for all  $f \in \bar{E}^-[0, 1]$  with  $\|f\|_\infty \leq \varepsilon_0$ . As  $Z_t$  may attain the value zero, we introduce the constant  $M$  to ensure finite values of the process.

Note that the copula process pertaining to the GPP  $\mathbf{Z}/U$  is in its upper tail given by the shifted standard GPP  $1 + \mathbf{V}$ , which satisfies the conditions of Theorem 3.5. We, therefore, obtain for the GPP process  $\mathbf{Z}/U$

$$\lim_{n \rightarrow \infty} \lim_{s \nearrow \omega(F)} \frac{FI_n(s)}{n} = \lim_{s \nearrow \omega(F)} E(S(s) \mid S(s) > 0) = \frac{1}{E(\max_{0 \leq t \leq 1} Z_t)}.$$

The mathematical tools from Section 2.3 enable also the computation of the (cumulative) expected shortfall corresponding to a stochastic process as defined below.

Let  $\mathbf{Y} = (Y_t)_{t \in [0, 1]}$  be a stochastic process in  $C[0, 1]$  with identical and continuous univariate marginal df  $F$  and put

$$I(s) = \int_0^1 (Y_t - s) 1(Y_t > s) dt.$$

The number  $I(s)$  can be interpreted as the total sum of excesses above the threshold  $s$ . The *expected shortfall* at level  $s$  pertaining to  $\mathbf{Y}$  is the total sum of excesses, given that there is at least one:

$$\text{ES}(s) := E(I(s) \mid S(s) > 0).$$

**Lemma 3.8.** *Let  $\mathbf{U} = (U_t)_{t \in [0,1]} = (F(Y_t))_{t \in [0,1]}$  be the copula process pertaining to  $\mathbf{Y}$ . Then we have*

$$\text{ES}(s) = \frac{\int_s^\infty 1 - F(x) dx}{1 - P\left(\sup_{t \in [0,1]} U_t \leq F(s)\right)}.$$

*Proof.* We have

$$\begin{aligned} E(I(s) \mid S(s) > 0) &= E\left(\int_0^1 (Y_t - s) 1(Y_t > s) dt \mid \int_0^1 1(Y_t > s) dt > 0\right) \\ &= E\left(\int_0^1 (Y_t - s) 1(Y_t > s) dt \mid \sup_{t \in [0,1]} Y_t > s\right) \\ &= \frac{E\left(\left(\int_0^1 (Y_t - s) 1(Y_t > s) dt\right) 1\left(\sup_{t \in [0,1]} Y_t > s\right)\right)}{P\left(\sup_{t \in [0,1]} Y_t > s\right)} \\ &= \frac{E\left(\int_0^1 (Y_t - s) 1(Y_t > s) dt\right)}{P\left(\sup_{t \in [0,1]} Y_t > s\right)}, \end{aligned}$$

where by Fubini's theorem

$$\begin{aligned} E\left(\int_0^1 (Y_t - s) 1(Y_t > s) dt\right) &= \int_0^1 E((Y_t - s) 1(Y_t > s)) dt \\ &= \int_0^1 \int_0^\infty 1 - P(Y_t - s \leq x) dx dt \\ &= \int_0^1 \int_0^\infty 1 - F(x + s) dx dt \\ &= \int_s^\infty 1 - F(x) dx \end{aligned}$$

and

$$\begin{aligned} P\left(\sup_{t \in [0,1]} Y_t > s\right) &= 1 - P\left(\sup_{t \in [0,1]} Y_t \leq s\right) \\ &= 1 - P\left(\sup_{t \in [0,1]} U_t \leq F(s)\right). \end{aligned}$$

□

Suppose in addition that the copula process  $\mathbf{U}$  is in the domain of attraction in the sense of condition (8) of a standard MSP with generator constant  $m$ . Then there exists a  $D$ -norm  $\|\cdot\|_D$  on  $C[0,1]$  with  $\|1\|_D = m$  such that

$$P(U_t \leq F(s), t \in [0,1]) = 1 - (1 - F(s)) \|1\|_D + o(1 - F(s))$$

as  $s \nearrow \omega(F)$ . The next result is, therefore, an obvious consequence of Lemma 3.8.

**Proposition 3.9.** *If in addition the copula process  $\mathbf{U}$  is in the domain of attraction of a standard MSP with generator constant  $m$ , then we obtain*

$$\text{ES}(s) = \frac{\int_s^\infty 1 - F(x) dx}{1 - F(s)} \left( \frac{1}{m} + o(1 - F(s)) \right)$$

as  $s \nearrow \omega(F)$ .

Proposition 3.9 precisely separates the contribution of the dependence structure of the stochastic process  $\mathbf{Y}$  on the expected shortfall as the threshold increases, which is  $1/\|\mathbf{1}\|_D = 1/m$ , from that of the marginal distribution, which is the first factor. In particular we obtain that the expected shortfall converges in  $[0, \infty)$  as  $s \nearrow \omega(F)$  if and only if  $\lim_{s \nearrow \omega(F)} \int_s^{\omega(F)} 1 - F(t) dt / (1 - F(s)) := c \in [0, \infty)$ . And in this case its limit is  $c/\|\mathbf{1}\|_D$ .

#### 4. SOJOURN TIME DISTRIBUTION

In this section we compute the asymptotic sojourn time distribution of such processes, which are in a certain neighborhood of a standard GPP. A standard MSP is a prominent example. In this setup we can replace the constant threshold  $s$  by a threshold function.

The sojourn time distribution of a standard GPP is easily computed as the following lemma shows. This distribution is independent of the threshold level  $s$ , which reveals another exceedance stability of a GPP. Note that we replace the constant threshold line  $s$  in what follows by a *threshold function*  $sf(t)$ , where  $f \in \bar{E}^-[0, 1]$  is fixed and  $s$  is the variable *threshold level*.

**Lemma 4.1.** *Let  $\mathbf{V}$  in  $\bar{C}^-[0, 1]$  be a standard GPP, i.e. there is an  $\varepsilon_0 > 0$  such that  $P(\mathbf{V} \leq g) = P(-U/\mathbf{Z} \leq g)$  for all  $g \in \bar{E}^-[0, 1]$  with  $\|g\|_\infty \leq \varepsilon_0$ , where  $U$  is uniformly on  $(0, 1)$  distributed and independent of the generator  $\mathbf{Z} = (Z_t)_{t \in [0, 1]}$ , which is continuous and satisfies  $0 \leq Z_t \leq m$ ,  $E(Z_t) = 1$ ,  $t \in [0, 1]$ , for some  $m \geq 1$ . Choose  $f \in \bar{E}^-[0, 1]$ . Then there is an  $s_0 > 0$  such that the sojourn time  $df H_f$  of  $\mathbf{V}$  above  $sf$  is given by*

$$\begin{aligned} & P \left( \int_0^1 1(V_t > sf(t)) dt > y \mid \int_0^1 1(V_t > sf(t)) dt > 0 \right) \\ &= \frac{\int_0^{m\|f\|_\infty} P \left( \int_0^1 1(|f(t)| Z_t > u) dt > y \right) du}{\int_0^{m\|f\|_\infty} P \left( \int_0^1 1(|f(t)| Z_t > u) dt > 0 \right) du} \\ &=: 1 - H_f(y), \quad 0 \leq y \leq 1, 0 < s \leq s_0, \end{aligned}$$

provided the denominator is greater than zero. Note that  $H_f(0) = 0$ ,  $H_f(1) = 1$ .

EXAMPLE 4.2. Any continuous df  $F$  on  $[0, 1]$  can occur as a sojourn time df. Take  $Z_t = 1$ ,  $0 \leq t \leq 1$ , which provides the case of complete dependence of the margins of the corresponding standard MSP  $\boldsymbol{\eta}$ . Choose a continuous df  $F : [0, 1] \rightarrow [0, 1]$  and put  $f(t) = F(t) - 1$ ,  $0 \leq t \leq 1$ . Then the sojourn time df equals  $F$ ,  $H_f(y) = F(y)$ ,  $y \in [0, 1]$ .

If we take, on the other hand,  $f(t) = -1$ ,  $t \in [0, 1]$ , then  $H_f$  has all its mass at 1, i.e.,  $H_f(y) = 0$ ,  $y < 1$ , and  $H_f(1) = 1$ . These examples show in particular that the sojourn time df  $H_f$  can be continuous as well as discrete.

*Proof.* The assertion is an immediate consequence of standard rules of integration together with conditioning on  $U = u$ :

$$\begin{aligned} & P \left( \int_0^1 1(V_t > sf(t)) dt > y \right) \\ &= P \left( \int_0^1 1(U < s|f(t)|Z_t) dt > y \right) \\ &= \int_0^1 P \left( \int_0^1 1(u < s|f(t)|Z_t) dt > y \right) du, \end{aligned}$$

where substituting  $u$  by  $su$  yields

$$\begin{aligned} &= s \int_0^{1/s} P \left( \int_0^1 1(|f(t)|Z_t > u) dt > y \right) du \\ &= s \int_0^{m\|f\|_\infty} P \left( \int_0^1 1(|f(t)|Z_t > u) dt > y \right) du \end{aligned}$$

if  $s \leq 1/(m\|f\|_\infty)$ . This implies the assertion.  $\square$

Next we will extend the preceding lemma to processes  $\boldsymbol{\xi}$  in  $\bar{C}^-[0, 1]$  which are in certain neighborhoods of a standard GPP  $\mathbf{V}$ . Precisely, we require that for a given function  $f \in \bar{E}_1^-[0, 1] := \{f \in \bar{E}^-[0, 1] : \|f\|_\infty \leq 1\}$

$$(9) \quad P(\xi_{t_i} > sf(t_i), 1 \leq i \leq k) = P(V_{t_i} > sf(t_i), 1 \leq i \leq k) + o(s)$$

for each set  $0 \leq t_1 < \dots < t_k \leq 1$ ,  $k \in \mathbb{N}$ , and

$$(10) \quad P(\boldsymbol{\xi} \leq sf) = P(\mathbf{V} \leq sf) + o(s)$$

as  $s \downarrow 0$ .

An example of a process satisfying conditions (9) and (10) is a standard MSP  $\boldsymbol{\eta}$ , which follows by Lemma 4.5 below together with equation (4). The next lemma follows from elementary computations.

**Lemma 4.3.** *For each standard GPP  $\mathbf{V}$  there exists  $s_0 > 0$  such that for  $0 \leq s \leq s_0$  and for each  $f \in \bar{E}^-[0, 1]$  with  $\|f\|_\infty \leq 1$*

(i)

$$P(\mathbf{V} \leq sf) = 1 - sE \left( \max_{t \in [0,1]} (|f(t)| Z_t) \right) = 1 - s \|f\|_D,$$

(ii)

$$P(\mathbf{V} > sf) = sE \left( \min_{t \in [0,1]} (|f(t)| Z_t) \right),$$

(iii)

$$P(V_{t_i} > sf(t_i), 1 \leq i \leq k) = sE \left( \min_{1 \leq i \leq k} (|f(t_i)| Z_{t_i}) \right)$$

for each set  $0 \leq t_1 < \dots < t_k \leq 1$ ,  $k \in \mathbb{N}$ .

The next result extends Lemma 4.1 to processes which satisfy condition (9) and (10).

**Proposition 4.4.** *Suppose that  $\xi \in \bar{C}^-[0, 1]$  has identical univariate margins and that it satisfies condition (9) as well as (10). Choose  $f \in \bar{E}_1^-[0, 1]$ . Then the asymptotic sojourn time distribution of  $\xi$ , conditional on the assumption that it is positive, is given by*

$$P \left( \int_0^1 1(\xi_t > sf(t)) dt > y \mid \int_0^1 1(\xi_t > sf(t)) dt > 0 \right) \rightarrow_{s \downarrow 0} 1 - H_f(y),$$

where the sojourn time  $df H_f$  is given in Lemma 4.1.

*Proof.* We establish this result by establishing convergence of characteristic functions. Put  $I_s := \int_0^1 1(\xi_t > sf(t)) dt$ ,  $s > 0$ . The characteristic function of the rv  $I_s$ , conditional on the event that it is positive, is

$$E(\exp(itI_s) \mid I_s > 0) = \frac{\int_{\{I_s > 0\}} \exp(itI_s) dP}{P(I_s > 0)}.$$

Note that  $0 \leq I_s \leq 1$ . By the dominated convergence theorem we have

$$\begin{aligned} \int_{\{I_s > 0\}} \exp(itI_s) dP &= \int_{\{I_s > 0\}} \sum_{k=0}^{\infty} \frac{(itI_s)^k}{k!} dP \\ &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \int_{\{I_s > 0\}} I_s^k dP \\ &= P(I_s > 0) + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \int_{\Omega} I_s^k dP \\ (11) \quad &= P(I_s > 0) + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} E(I_s^k). \end{aligned}$$

From condition (10) we obtain

$$\begin{aligned} P(I_s > 0) &= 1 - P(I_s = 0) \\ &= 1 - P(\xi \leq sf) \end{aligned}$$



$$\begin{aligned}
&= 1 - P(\mathbf{V} \leq sf) + o(s) \\
(12) \quad &= s \left( E \left( \max_{t \in [0,1]} |f(t)Z_t| \right) + o(1) \right)
\end{aligned}$$

as  $s \downarrow 0$ .

From Fubini's theorem we obtain for  $k \in \mathbb{N}$

$$\begin{aligned}
E(I_s^k) &= E \left( \left( \int_0^1 1(\xi_t > sf(t)) dt \right)^k \right) \\
&= E \left( \int_0^1 \dots \int_0^1 \prod_{i=1}^k 1(\xi_{t_i} > sf(t_i)) dt_1 \dots dt_k \right) \\
&= \int_0^1 \dots \int_0^1 E \left( \prod_{i=1}^k 1(\xi_{t_i} > sf(t_i)) \right) dt_1 \dots dt_k \\
&= \int_0^1 \dots \int_0^1 P(\xi_{t_i} > sf(t_i), 1 \leq i \leq k) dt_1 \dots dt_k.
\end{aligned}$$

We have by condition (9)

$$P(\xi_{t_i} > sf(t_i), 1 \leq i \leq k) \leq P(\xi_{t_1} > -s) = P(\xi_0 > -s) = P(V_0 > -s) + o(s)$$

uniformly for  $t_1, \dots, t_k \in [0, 1]$  and, thus,  $P(\xi_{t_i} > sf(t_i), 1 \leq i \leq k) / s$  is uniformly bounded. Condition (9) together with the dominated convergence theorem now implies

$$\begin{aligned}
\frac{E(I_s^k)}{s} &= \int_0^1 \dots \int_0^1 \frac{P(\xi_{t_i} > sf(t_i), 1 \leq i \leq k)}{s} dt_1 \dots dt_k \\
(13) \quad &\rightarrow_{s \downarrow 0} \int_0^1 \dots \int_0^1 E \left( \min_{1 \leq i \leq k} |f(t_i)Z_{t_i}| \right) dt_1 \dots dt_k.
\end{aligned}$$

From equations (11)-(13) we obtain

$$\begin{aligned}
&\int_{\{I_s > 0\}} \exp(itI_s) dP \\
&= s(1 + o(1)) \left( E \left( \max_{t \in [0,1]} |f(t)Z_t| \right) \right. \\
&\quad \left. + \sum_{k=1}^n \frac{(it)^k}{k!} \left( \int_0^1 \dots \int_0^1 E \left( \min_{1 \leq i \leq k} |f(t_i)Z_{t_i}| \right) dt_1 \dots dt_k \right) \right) \\
&\quad + \sum_{k=n+1}^{\infty} \frac{(it)^k}{k!} E(I_s^k),
\end{aligned}$$

where  $n \in \mathbb{N}$  is chosen such that for a given  $\varepsilon > 0$  we have  $\sum_{k=n+1}^{\infty} 1/k! \leq \varepsilon$ . As  $I_s \in [0, 1]$ , we obtain

$$E(I_s^k) \leq E(I_s)$$

$$\begin{aligned}
&= \int_0^1 P(\xi_t > sf(t)) dt \\
&= \int_0^1 P(\xi_0 > sf(t)) dt \\
&\leq P\left(\xi_0 > s \inf_{t \in [0,1]} f(t)\right) \\
&= s \inf_{t \in [0,1]} |f(t)| + o(s)
\end{aligned}$$

by condition (9) and, thus,

$$\begin{aligned}
&\int_{\{I_s > 0\}} \exp(itI_s) dP \\
&= s(1 + o(1)) \left( E \left( \max_{t \in [0,1]} |f(t)Z_t| \right) \right. \\
&\quad \left. + \sum_{k=1}^n \frac{(it)^k}{k!} \left( \int_0^1 \dots \int_0^1 E \left( \min_{1 \leq i \leq k} |f(t_i)Z_{t_i}| \right) dt_1 \dots dt_k \right) + O(\varepsilon) \right)
\end{aligned}$$

as  $s \downarrow 0$ . Since  $\varepsilon > 0$  was arbitrary we obtain

$$\begin{aligned}
&\lim_{s \downarrow 0} \frac{\int_{\{I_s > 0\}} \exp(itI_s) dP}{P(I_s > 0)} \\
&= 1 + \frac{\sum_{k=1}^{\infty} \frac{(it)^k}{k!} \left( \int_0^1 \dots \int_0^1 E \left( \min_{1 \leq i \leq k} |f(t_i)Z_{t_i}| \right) dt_1 \dots dt_k \right)}{E \left( \max_{t \in [0,1]} |f(t)Z_t| \right)} \\
&=: \varphi(t), \quad t \in \mathbb{R}.
\end{aligned}$$

An inspection of the preceding arguments shows that  $\varphi$  is the characteristic function of the sojourn time df  $H_f$ , which completes the proof.  $\square$

We conclude this section by showing that a standard MSP  $\boldsymbol{\eta}$  satisfies condition (9) and, thus, Proposition 4.4 applies. Note that condition (10) follows from (4) and Taylor expansion of exp.

**Lemma 4.5.** *Let  $\boldsymbol{\eta}$  be a standard MSP with generator  $\mathbf{Z}$ . Then we obtain for  $f \in \bar{E}^- [0, 1]$*

$$P(\eta_{t_i} > sf(t_i), 1 \leq i \leq k) = sE \left( \min_{1 \leq i \leq k} (|f(t_i)| Z_{t_i}) \right) + o(s)$$

as  $s \downarrow 0$  for any set  $0 \leq t_1 < \dots < t_k \leq 1$ ,  $k \in \mathbb{N}$ .

*Proof.* The inclusion-exclusion theorem yields

$$P \left( \bigcap_{i=1}^k \{\eta_{t_i} > sf(t_i)\} \right)$$

$$\begin{aligned}
&= 1 - P\left(\bigcup_{i=1}^k \{\eta_{t_i} \leq sf(t_i)\}\right) \\
&= 1 - \sum_{\emptyset \neq T \subset \{1, \dots, k\}} (-1)^{|T|-1} P\left(\bigcap_{i \in T} \{\eta_{t_i} \leq sf(t_i)\}\right) \\
&= 1 - \sum_{\emptyset \neq T \subset \{1, \dots, k\}} (-1)^{|T|-1} \exp\left(-sE\left(\max_{i \in T} (|f(t_i)| Z_{t_i})\right)\right) \\
&=: 1 - H(s) \\
&= H(0) - H(s),
\end{aligned}$$

where the function  $H$  is differentiable and, thus,

$$\begin{aligned}
\lim_{s \downarrow 0} \frac{P(\eta_{t_i} > sf(t_i), 1 \leq i \leq k)}{s} &= -\lim_{s \downarrow 0} \frac{H(s) - H(0)}{s} \\
&= -H'(0) \\
&= \sum_{\emptyset \neq T \subset \{1, \dots, k\}} (-1)^{|T|-1} E\left(\max_{i \in T} (|f(t_i)| Z_{t_i})\right) \\
&= E\left(\min_{i \in T} (|f(t_i)| Z_{t_i})\right),
\end{aligned}$$

since  $\sum_{\emptyset \neq T \subset \{1, \dots, k\}} (-1)^{|T|-1} \max_{i \in T} a_i = \min_{1 \leq i \leq k} a_i$  for arbitrary numbers  $a_1, \dots, a_k \in \mathbb{R}$ , which can be seen by induction.  $\square$

## 5. EXCURSION TIME

The considerations in the previous section enable us also to compute the limit distribution of the excursion time above the threshold  $sf$  of a process  $\mathbf{X}$  in  $\bar{C}^- [0, 1]$ , which is in a neighborhood of a standard GPP. Precisely, we require the following condition. Choose  $0 \leq a \leq b \leq 1$ , and denote by  $\bar{C}^- [a, b]$  the set of continuous functions  $f : [a, b] \rightarrow (-\infty, 0]$ . We suppose that for  $f \in \bar{C}^- [a, b]$

$$(14) \quad P(X_t > sf(t), t \in [a, b]) = P(V_t > sf(t), t \in [a, b]) + o(s)$$

as  $s \downarrow 0$ , where  $\mathbf{V} = (V_t)_{t \in [0, 1]}$  is a standard GPP. Note that

$$(15) \quad P(V_t > sf(t), t \in [a, b]) = sE\left(\min_{a \leq t \leq b} (|f(t)| Z_t)\right) + o(s), \quad s \in (0, s_0),$$

and that we allow the case  $a = b$ . We do not require  $\mathbf{X}$  to have identical marginal distributions.

A standard MSP  $\boldsymbol{\eta}$  satisfies condition (14), see Aulbach et al. [2]. Another example is the following class of processes. Substitute the rv  $U$  in the GPP  $\mathbf{V} =$

$(\max(-U/Z_t, M))_{t \in [0,1]}$  by a rv  $W \geq 0$ , which is independent of  $\mathbf{Z}$  as well and whose df  $H$  satisfies

$$H(x) = x + o(x), \quad \text{as } x \rightarrow 0.$$

The standard exponential df  $F(x) = 1 - \exp(-x)$ ,  $x > 0$ , is a typical example. Then the process

$$\mathbf{X} := \left( \max \left( -\frac{W}{Z_t}, M \right) \right)_{t \in [0,1]}$$

satisfies condition (15) as well.

The *remaining excursion time* above  $sf$  of the process  $\mathbf{X}$  with inspection point  $t_0 \in [0, 1)$  is the remaining time that the process spends above  $sf$ , under the condition that  $X_{t_0} > sf(t_0)$ , i.e., it is defined by

$$\tau_{t_0}(s) := \sup \{L \in (0, 1 - t_0] : X_t > sf(t), t \in [t_0, t_0 + L]\}$$

under the condition that  $X_{t_0} > sf(t_0)$ .

**Proposition 5.1.** *Suppose that  $\mathbf{X}$  in  $\bar{C}^-[0, 1]$  satisfies condition (14). Then we have for  $u \in [0, 1 - t_0)$  and  $f \in \bar{C}^-[a, b]$  with  $f(t_0) < 0$*

$$\lim_{s \downarrow 0} P(\tau_{t_0}(s) > u \mid X_{t_0} > sf(t_0)) = \frac{E(\min_{t_0 \leq t \leq u} (|f(t)| Z_t))}{|f(t_0)|}.$$

*Proof.* We have for  $u \in [0, 1 - t_0)$

$$\begin{aligned} P(\tau_{t_0}(s) > u \mid X_{t_0} > sf(t_0)) &= \frac{P(X_t > sf(t), t \in [t_0, t_0 + u])}{P(X_{t_0} > sf(t_0))} \\ &= \frac{P(V_t > sf(t), t \in [t_0, t_0 + u]) + o(s)}{P(V_{t_0} > sf(t_0)) + o(s)} \\ &= \frac{E(\min_{t_0 \leq t \leq t_0 + u} (|f(t)| Z_t))}{|f(t_0)|} + o(1) \end{aligned}$$

as  $s \downarrow 0$ . □

The *asymptotic* remaining excursion time  $T_{t_0}$ , as  $s \downarrow 0$ , with inspection point  $t_0 \in [0, 1)$  has, consequently, the continuous df

$$P(T_{t_0} \leq u) = 1 - \frac{E(\min_{t_0 \leq t \leq t_0 + u} (|f(t)| Z_t))}{|f(t_0)|}$$

for  $0 \leq u < 1 - t_0$ , and possibly positive mass at  $1 - t_0$ :

$$P(T_{t_0} = 1 - t_0) = \frac{E(\min_{t_0 \leq t \leq 1} (|f(t)| Z_t))}{|f(t_0)|}.$$

Its expected value is, therefore, given by

$$\begin{aligned} E(T_{t_0}) &= \int_0^{1-t_0} P(T_{t_0} > u) du \\ &= \frac{1}{|f(t_0)|} \int_0^{1-t_0} E \left( \min_{t_0 \leq t \leq t_0 + u} (|f(t)| Z_t) \right) du \end{aligned}$$

$$= \frac{1}{|f(t_0)|} E \left( \int_{t_0}^1 \min_{t_0 \leq t \leq u} (|f(t)| Z_t) du \right).$$

## ACKNOWLEDGMENT

The authors are grateful to two anonymous reviewers and the associate editor for their careful reading of the manuscript and their constructive remarks, from which the paper has benefitted a lot.

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UNIVERSITY OF WÜRZBURG  
INSTITUTE OF MATHEMATICS  
EMIL-FISCHER-STR. 30  
97074 WÜRZBURG  
GERMANY  
HOFMANN.MARTIN@MATHEMATIK.UNI-WUERZBURG.DE